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The Spin-Statistics Theorem*

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A derivation of the connection between spin and statistics is obtained for spin 0, $\frac{1}{2}$, and 1 fields with arbitrary local interactions. The basis used is the Schwinger action principle, whose assumptions are specified; they include neither positive energy spectrum nor *TCP* invariance. The connection can be obtained without either of these two extra requirements in most cases. The remaining cases are characterized by non-*TCP* invariant *free* Lagrangians and nonpositive definite free-particle energies. Commutation relations among different fields are also briefly discussed by means of the action principle.

I. INTRODUCTION

After Pauli's original derivation of the connection between spin and statistics,[1] a number of deductions of this result, using various sets of assumptions, have been presented. These derivations have removed the restriction to noninteracting fields involved in the initial proof. Thus, the work of Schwinger[2] employed *TCP* invariance, while the approach exemplified by Burgoyne and Luders and Zumino[3] assumed the existence of a vacuum state representing the lowest energy of the system. The latter postulate ensures certain analyticity properties of vacuum expectation values, which together with other quite general requirements leads to the connection. This elegant method provides a direct generalization of Pauli's proof to coupled fields. However, in the uncoupled case, the existence of a vacuum state was only invoked to forbid Bose quantization of charged half-integral spin fields. It is therefore of interest to see to what extent one may avoid such additional postulates as the energy requirement and *TCP* invariance for coupled fields.

In this note, we shall start from the Schwinger action principle,[4] which considers only systems with a local Lagrangian, but does not demand a vacuum state. An explicit statement of the principle and its assumptions is given; as in all other derivations, we require that the Hilbert space metric be positive-definite and consider only the possibilities of commutativity or anticommutativity. We shall divide the problem into four parts according to whether the field is neutral or charged and has integral or half-integral spin. Our explicit derivation will be made for 0, $\frac{1}{2}$, and 1 spins only.[5] We shall show that in all but the massless neutral spin $\frac{1}{2}$ case and charged spin $\frac{1}{2}$ case, neither the vacuum assumption nor *TCP* invariance is needed, irrespective of interactions. In these cases, a "wrong" connection leads to purely algebraic inconsistencies, reminiscent of those originally found by Pauli for free integral spin fields.

II. THE ACTION PRINCIPLE FRAMEWORK

The action principle,[4] upon which our treatment of the spin-statistics connection is based, requires in its derivation a number of specific postulates. We therefore first list these and discuss briefly their nature. The Appendix contains a more complete treatment.

I. The conventional Hilbert space interpretation of quantum mechanics, with positive-definite metric holds.

II. The system is invariant with respect to the proper orthochronous inhomogeneous Lorentz group.

III. The characterization of a state at a given time and the equations of its time development are local in time (i.e., we are dealing with a local field theory).

In order to discuss the remaining postulates, we introduce some notation. Let $\delta\langle a_1t_1 \mid a_2t_2\rangle$ be the change of a transformation function[6] under infinites-

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[1] W. Pauli, Phys. Rev. **58**, 716 (1940).

[2] J. Schwinger, Phys. Rev. **82**, 914 (1951)., Proc. Natl. Acad. Sci. U. S. **44**, 223, 617 (1958).

[3] N. Burgoyne, Nuovo cimento **8**, 607 (1958); G. Luders and B. Zumino, Phys. Rev. **110**, 1450 (1958).

[4] J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1953).

[5] From the structure of the proof, we expect that the generalization to higher spin fields should be feasible.

[6] For simplicity, we are assuming here that out operators and states are defined at a fixed time. A more general treatment, in terms of space-like surfaces can, of course, be given.

imal transformations which alter the complete set $\{A\}$ at time t (here $A \mid a_1t\rangle = a_1 \mid a_1t\rangle$) and move the system in time: $t \to t + \delta t$. These changes are unitary transformations on the basis vectors according to I. Defining the infinitesimal operator δW_{12} by $\delta\langle a_1t_1 \mid a_2t_2\rangle \equiv i\langle a_1t_1 \mid \delta W_{12} \mid a_2t_2\rangle$, (so that δW_{12} is necessarily Hermitian) we postulate that

IV (a). There exists a finite operator W_{12} such that a unique set of variations on its operator form yield δW_{12} for the classes of transformations considered above.

One can show (see Appendix) that W_{12} has the form of a space-time integral over the region between t_1 and t_2,

$$W_{12} = \int_{t_2}^{t_1} d^4x \mathfrak{L}(x)$$

(with the scalar function $\mathfrak{L}$ Hermitian). We further postulate that

$$\text{IV(b)} \quad \mathfrak{L}(x) = \chi A^\mu \partial_\mu \chi - \partial_\mu \chi A^\mu \chi - \mathfrak{H}(\chi) + \partial_\mu W^\mu(\chi),$$

where χ is a column symbol whose components are all the field variables and A^μ are constant numerical matrices in that space.[7]

From postulate III, $\mathfrak{H}$ is a local function, while Hermiticity of $\mathfrak{L}$ requires the A^μ to be skew-Hermitian and $\mathfrak{H}$ and W^μ Hermitian. The generalized Kemmer form assumed above for $\mathfrak{L}$ is no essential restriction, since any local field system obeying second order equations with at most first derivative coupling can be described by such a Lagrangian,[8] with $\mathfrak{H}(\chi)$ independent of derivatives of χ.

In varying W_{12} to yield the required $\delta\langle a_1t_1 \mid a_2t_2\rangle$ it can be shown (see Appendix) that variations of the time $t \to t + \Delta t$ and of the field variables $\chi \to \chi + \bar{\delta}\chi$ must be made throughout the space-time region. The relation of these changes to specific unitary transformations carried out on the transformation function must be specified in order to give meaning to the basic postulate IV(a). We therefore assume

V(a) If δt is the time translation carried out on the transformation function, then $\Delta t = \alpha(t)\, \delta t$, α is a c number, i.e., Δt vanishes when no unitary transformations corresponding to pure time motion are made.

V(b) The variation $\bar{\delta}\chi_\alpha$ (of the field variable χ_α) in $\mathfrak{L}$ either commutes or anticommutes with a given field variable χ_β.

The most general *a priori* possibility for Δt would be the form $\Delta t = \alpha(t)\, \delta t + \delta\beta$, where $\delta\beta$ does not vanish when a change of basis at fixed time is made, i.e., when $\delta t = 0$. The requirement V(a) is thus that $\delta\beta = 0$. It can then be shown that $\alpha = 1$ and the not unexpected result $\Delta t = \delta t$ holds. Postulate V(b) is a condition only on the operator nature of $\bar{\delta}\chi$. Some rule on the operator character of $\bar{\delta}\chi$ is needed to obtain well-defined Lagrange equations.[9] As will be seen, V(b) leads to either commutation or anticommutation relations holding between field variables and hence is similar to the assumptions commonly made in other discussions of spin and statistics. At this stage, we may compare our set of postulates with other starting points of field theory. Assumptions I, II, and V(b) are postulates conventionally included in other treatments of field theory, while IV and V(a) are characteristic of the action principle. Postulates III and IV(b) (which limit us to local fields) and a final requirement VI, to be introduced below, are invoked in all discussions of local field theory.

The above postulates allow us to obtain the Lagrange equations from W_{12} (see discussion in the Appendix). One finds

$$(-\bar{\delta}\chi A^\mu \partial_\mu \chi + \partial_\mu \chi A^\mu \bar{\delta}\chi) - m(\bar{\delta}\chi B \chi + \chi B \bar{\delta}\chi) - \bar{\delta}\chi\, \partial\mathfrak{H}_I/\partial\chi = 0, \quad (2.1a)$$

where we have written $\mathfrak{H} = m\chi B\chi + \mathfrak{H}_I(\chi)$ to exhibit a possible mass term for the field. The symbol $\bar{\delta}\chi\, \partial\mathfrak{H}_I/\partial\chi$ stands for $\mathfrak{H}_I(\chi + \bar{\delta}\chi) - \mathfrak{H}_I(\chi)$ and the matrix B is necessarily Hermitian. Explicit equations of motion can now be obtained by using the commutation relations obeyed by the $\bar{\delta}\chi$ to move them all to one side in Eq. (2.1a), and equating their coefficients to zero. For each field of given spin we will see that the A^μ are all either symmetric or antisymmetric. In the symmetric (antisymmetric) case, one will obtain equations with space-time derivatives only if $\bar{\delta}\chi$ is taken to anticommute (commute). The "wrong" choice of commutation relations for $\bar{\delta}\chi$ then reduces the content of the Lagrange equations to $\bar{\delta}\chi\, \partial\mathfrak{H}/\partial\chi = 0$ [since the first parenthesis in Eq. (2.1a) vanishes]. This is either an identity $(0 = 0)$ or an algebraic relation (constraint) among the χ's at any time (which may or

[7] We use the metric $\eta_{\mu\nu} = \text{diag}(1, 1, 1, -1)$, with Latin indices varying over 1, 2, 3, Greek over 1, 2, 3, 0, and natural units: $\hbar = 1 = c$.

[8] If higher derivatives than the first had been present in the interaction, one could have adjoined the derivatives as new variables to χ to reach first-order form; however, the χ space is then not irreducible, which greatly complicates the analysis. We do not consider such couplings in this work.

[9] This restriction also guarantees that the quantum Lagrange equations resemble the classical ones in form; more complicated relations between variations and fields would lose this feature.

may not be consistent). In either case, no true Lagrange equations arise from use of the "wrong" choice. If the "right" choice is taken, one gets the standard Kemmer-Dirac equations of motion with local interactions:

$$A^{\mu}\,\partial_{\mu}\chi + mB\chi + \tfrac{1}{2}\,\partial\mathcal{H}_I/\partial\chi = 0 \qquad (2.1\text{b})$$

From our postulates, one finds (see Appendix) that the generator of the unitary transformation for time translation ($\mathcal{U} = 1 + iG_t$) is

$$G_t = -H\,\delta t = -\int d^3r[\tfrac{1}{2}(-\chi A^i\,\partial_i\chi + \partial_i\chi A^i\chi) + \mathcal{H}]\,\delta t, \qquad (2.2\text{a})$$

where H is the usual field Hamiltonian. The effect of G_t on χ is to translate it in time by an amount δt, i.e.,

$$[\chi, G_t] = i\dot{\chi}\,\delta t. \qquad (2.2\text{b})$$

Equation (2.2b) is the Heisenberg equation of motion. The generator G' of arbitrary unitary transformations at a fixed time may be shown to be

$$G' = G_\chi + \int d^3r\;\bar{\delta}\chi\,(\partial W^0/\partial\chi) \qquad (2.3)$$

where

$$G_\chi \equiv \int d^3r\tfrac{1}{2}(\chi A^0\,\bar{\delta}\chi - \bar{\delta}\chi A^0\chi) \qquad (2.4)$$

and $\bar{\delta}\chi\,(\partial W^0/\partial\chi)$ stands for $W^0(\chi + \bar{\delta}\chi) - W^0(\chi)$. A special case of importance is the choice $W^0 = 0$, i.e., $G' = G_\chi$. The effect of G_χ on χ is to change it by an amount proportional to $\bar{\delta}\chi$. One may express this in general by writing

$$[\chi, G_\chi] = \tfrac{1}{2}if\,\bar{\delta}\chi \qquad (2.5)$$

where f is an unknown operator. Our final postulate[9a] is

VI. f is a c number.

The "wrong" choice for $\bar{\delta}\chi$ reduces G_χ to zero identically, and as we have seen, also fails to yield valid Lagrange equations. Hence, we may drop this empty possibility and retain only the "right" choice of $\bar{\delta}\chi$ in accordance with the symmetry character of A^{μ}. In this case, f is necessarily unity, due to the consistency requirement between the Lagrange and Heisenberg equations. This result arises from the following considerations. The effects of the generators G_t and G_χ are given by Eqs. (2.2b) and (2.5). These relations are not independent, but are subject to the important consistency requirement that the effect of G_t on G_χ agree with that of G_χ on G_t. We have on the one hand from the action of G_χ on χ that

$$-i[G_\chi, H[\chi]] = \int d^3r[T^{00}(\chi - \tfrac{1}{2}f\,\bar{\delta}\chi) - T^{00}(\chi)], \qquad (2.6\text{a})$$

where T^{00} is the energy density. On the other hand, we may evaluate (2.6a) through the effect of G_t on χ:

$$-i[G_\chi, H] = -i\int d^3r[\chi A^0\,\bar{\delta}\chi, H] = \int d^3r A^0\dot{\chi}\,\bar{\delta}\chi. \qquad (2.6\text{b})$$

Here we have used the fact (shown in the Appendix) that $\bar{\delta}\chi$ commutes with H. Equations (2.6) express $A^0\dot{\chi}$ as a function of the χ's. However, $A^0\dot{\chi}$ is also specified through the Lagrange equations (2.1b) by use of which the right member of Eq. (2.6b) may be replaced by $\int d^3r[T^{00}(\chi - \frac{1}{2}\bar{\delta}\chi) - T^{00}(\chi)]$, and so $f = 1$ follows from VI. In the Appendix, the more general consistency requirements between an arbitrary G' and G_t are examined and found to allow $f = 1$.

The equal-time commutation relations among χ's are established from Eq. (2.5), which reads explicitly

$$\tfrac{1}{2}\int d^3r[\chi', \chi A^0\,\bar{\delta}\chi - \bar{\delta}\chi A^0\chi] = \tfrac{1}{2}i\,\bar{\delta}\chi', \qquad \chi' \equiv \chi(\mathbf{r}', t). \qquad (2.7)$$

If A^0 is antisymmetric, we have seen that $A^0\bar{\delta}\chi$ commutes with χ, so that Eq. (2.7) becomes

$$\int d^3r[\chi', \chi]A^0\,\bar{\delta}\chi = \tfrac{1}{2}i\,\bar{\delta}\chi' \qquad (2.8)$$

For nonsingular A^0, it then follows that

$$[\chi', \chi] = \tfrac{1}{2}iA_0^{-1}\,\delta^3(\mathbf{r} - \mathbf{r}') \qquad (2.9\text{a})$$

while if A^0 is singular, one cannot deduce a simple commutation relation between χ and χ'. A singular A^0 implies the existence of constraints in the theory.[10] Aside from fields, such as the electromagnetic one, which possess a gauge group, this situation presents no difficulty: as we shall see for the explicit cases to be treated, the $A^0\chi$ turn out to be all the independent field variables and one

[9a] *Note added in proof.* This assumption is actually derivable from the previous postulates. See, "Note on Uniqueness of Canonical Commutation Relations," J. Math. Phys. (to be published).

[10] See for example, reference 4.

gets the complete set of commutators from Eq. (2.8):

$$[A^0\chi', A^0\chi] = \tfrac{1}{2}iA^0\,\delta^3(\mathbf{r}-\mathbf{r}') \qquad (2.9b)$$

The electromagnetic field must be treated separately (see end of Sec. IV below). For symmetric nonsingular A^0, the result corresponding to Eq. (2.9a) is

$$\{\chi', \chi\} = \tfrac{1}{2}iA_0^{-1}\,\delta^3(\mathbf{r}-\mathbf{r}'). \qquad (2.10)$$

The singular case does not arise for spin $\frac{1}{2}$. We have thus found that all fields obey commutations or anticommutation relations as a result of postulate $V(a)$, and that the equal-time relations are c numbers due to postulate VI.

It should be noted that the equal-time commutation relations (2.9, 2.10) have been obtained purely from the kinetic part of the Lagrangian. The symmetry character of B and the nature of $\mathcal{H}_I$ were not involved. We also mention that for each field, a complete set of equal time commutation relations were found; thus, in the charged scalar case for example, we shall get not only $[\varphi', \varphi^+]$ and $[\varphi', \pi]$ but also $[\varphi', \varphi]$. It is really the last commutator which is the "statistics" part of the theorem, i.e., the one which allows or forbids more than one particle per state. Without a complete particle interpretation and a derivation of $[\varphi, \varphi']$-like relations, the theorem is not fully established.[11]

III. NEUTRAL SPIN $\frac{1}{2}$

We begin with the special case of the Majorana field with nonvanishing mass. The field equations read here

$$A^\mu\,\partial_\mu\chi + mB\chi + (\tfrac{1}{2})\,\partial\mathcal{H}_I/\partial\chi = 0. \qquad (3.1)$$

The matrix B is necessarily nonsingular in order that a Dirac equation of the form (3.1) exist. The Dirac γ^μ are then formed from A^μ, B according to

$$\gamma^\mu = iB^{-1}A^\mu, \qquad \{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \qquad (3.2)$$

and the A^μ must also be nonsingular, since the γ^μ are nonsingular. In general, $A^\mu = a^\mu + s^\mu$ where s^μ and a^μ are, respectively, symmetric and antisymmetric. The variation $\bar{\delta}\chi_\alpha$ is assumed either to commute or anticommute[12] with a given χ_β; hence the argument in Sec. II implies that in the former case only the terms with a^μ, and in the latter case only those with s^μ, will survive in the Lagrange equations and in the generator. We may therefore examine the cases $A^\mu = s^\mu$, a^μ separately. Treating first $A^\mu = s^\mu$, (so that $s^{\mu *} = -s^\mu$), we note that all χ_α anticommute by Eq. (2.10). Thus the $\chi B\chi$ term in $\mathcal{L}$ will vanish unless B is antisymmetric, and so without loss of generality, we require $\tilde{B} = -B$, so that $B^* = -B$ as B is Hermitian. To establish the fact that this is the Majorana field, we show that one can build up the Dirac algebra from s^μ and an antisymmetric B. This is accomplished by the choice $B = \gamma^0$, $s^\mu = -i\gamma^0\gamma^\mu$ where γ^μ are the usual Dirac matrices in the Majorana representation. With this choice, Eq. (2.10) becomes the standard anticommutation relation for the Majorana field.

We show next that the opposite symmetry assumption, i.e., $A^\mu = a^\mu$ and consequently $\tilde{B} = B$, $B^* = B$, which implies Bose relations by Eq. (2.9a), is not possible. In particular we now show that Eq. (3.2) cannot be satisfied. From the assumed symmetry properties of a^μ, B, and the Majorana γ^μ ($\tilde{\gamma}^0 = -\gamma^0$, $\tilde{\gamma}^i = \gamma^i$) one finds

$$[B, \gamma^0] = 0 = \{B, \gamma^i\}. \qquad (3.3)$$

Since the neutral spin $\frac{1}{2}$ field is a 4×4 realization of the Dirac algebra, B must be constructed from the 16 Dirac matrices; Eqs. (3.3) require $B = \eta\gamma^0$ (η a number). However, then $\tilde{B} = -B$, which contradicts the assumed symmetry of B.

The massless case needs separate treatment, since there is then no B in $\mathcal{L}$. One is therefore free to investigate the possibility of adjoining any matrix b to the A^μ such that $\gamma^\mu = ib^{-1}A^\mu$. Again, the assumption that $A^\mu = s^\mu$ (the normal case) clearly leads to the correct connection, as in the $m \neq 0$ case, with $b = \gamma^0$ (the choice $\tilde{b} = b$ is impossible here). The other possibility is $A^\mu = a^\mu$, which leads to Bose quantization, for arbitrary $b = b^s + b^a$($\tilde{b}^s = b^s$, $\tilde{b}^a = -b^a$). The symmetry properties of γ^μ then imply that

$$[b^s, \gamma^0] + \{b^a, \gamma^0\} = 0 \qquad (3.6a)$$

$$\{b^s, \gamma^i\} + [b^a, \gamma^i] = 0 \qquad (3.6b)$$

from which it follows that

$$b^s = 0, \qquad b^a = i\eta\gamma^0\gamma^5 \qquad (3.7a)$$

$$a^\mu = \eta\gamma^5\gamma^0\gamma^\mu \qquad (3.7b)$$

where η is a real number and $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$. The commutation relations (2.9a) now read

$$[\chi'_\alpha, \chi_\beta] = -(i/2\eta)\gamma^5_{\alpha\beta}\,\delta^3(\mathbf{r}-\mathbf{r}') \qquad (3.8)$$

[11] G. Feinberg has pointed out that the Burgoyne derivation does not establish the connection for the $[\phi, \phi']$ relations in the charged case. These relations have recently been established within the framework of reference 3, however, by G. F. Dell' Antonio and by A. S. Wightman (private communication to G. Feinberg). Dell' Antonio's derivation appears in Ann. Phys. **16**, 153, (1961).

[12] By Lorentz covariance, all components $\bar{\delta}\chi_\alpha$ of, say, a spinor will either commute with a given χ_β or all will anticommute.

while the action and Hamiltonian are given by

$$I = \int \mathcal{L}\, d^4x = \int [\chi\eta\gamma^5\gamma^0\gamma^\mu\, \partial_\mu\chi - \mathcal{H}_I]\, d^4x \qquad (3.9a)$$

$$H = \int [-\eta\chi\gamma^5\gamma^0\gamma^i\, \partial_i\chi + \mathcal{H}_I]\, d^3r. \qquad (3.9b)$$

The field with the properties (3.8), (3.9) has no manifest inconsistencies in that the Lagrange and Heisenberg equations agree. However, it possesses a number of strange features. First, the free Hamiltonian is not a positive-definite operator, and states with opposite helicity have opposite signs of energy. Second, the theory is invariant under neither P nor TCP. The Hermitian nature of the field means that C invariance holds trivially and so the P nonconservation cannot be compensated by C. (By contrast, the usual massless Majorana field with $I = \int \chi\gamma^0\gamma^\mu\, 1/i\, \partial_\mu\chi\, d^\mu x$ conserves both P and TCP, of course.)[13]

To eliminate this case of wrong connection, we may therefore invoke the TCP requirement. Though Pauli[1] did not originally consider such a field, one would have to make the same demand within his framework to avoid it. (Alternately, for free fields, the vacuum state condition would also be sufficient.)[14]

IV. NEUTRAL SPIN 0, 1 FIELDS

We consider next neutral integral spin fields, with or without mass. The zero spin Lagrangian in first-order form is

$$\mathcal{L} = \tfrac{1}{4}\{\phi, \partial_\mu\phi^\mu\} - \tfrac{1}{4}\{\partial_\mu\phi, \phi^\mu\} - \tfrac{1}{2}(\mu^2\phi^2 - \phi_\mu\phi^\mu) - \mathcal{H}_I, \qquad (4.1)$$

which clearly gives the usual field equations upon independent variation of ϕ and ϕ_μ (the anticommutators in $\mathcal{L}$ are needed to preserve its Hermitian character).[15] The Kemmer form (2.1a) of $\mathcal{L}$ is obtained by introducing the vector $\chi \equiv (\phi, \phi_\mu)$, so that A^μ and B are 5×5 matrices. In particular

$$A^0 = \tfrac{1}{2}\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} = -\tilde{A}^0, \qquad \alpha \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad (4.2)$$

and B is diagonal, with elements $\tfrac{1}{2}(\mu^2, 1, -1, -1, -1)$.

For the neutral spin one case, the Lagrangian reads

$$\mathcal{L} = \tfrac{1}{4}\{\phi_\nu, \partial_\mu G^{\mu\nu}\} - \tfrac{1}{4}\{\partial_\mu\phi_\nu, G^{\mu\nu}\} - \tfrac{1}{2}(\mu^2\phi_\mu\phi^\mu - \tfrac{1}{2}G_{\mu\nu}G^{\mu\nu}) - \mathcal{H}_I \qquad (4.3)$$

where $G^{\mu\nu} = -G^{\nu\mu}$ and ϕ_μ are to be varied independently. Again, in terms of $\chi \equiv (\phi_1, G_{01}, \phi_2, G_{02}, \phi_3, G_{03}, \phi_0, G_{ij})$, Eq. (4.3) has the form (2.1a) with $\tilde{A}^\mu = -A^\mu$, $\tilde{B} = B$, the 10×10 matrix A^0 being

$$A^0 = \tfrac{1}{2}\begin{bmatrix} \alpha & & & & \\ & \alpha & & 0 & \\ & & \alpha & & \\ & 0 & & 0 & \\ & & & & 0 \end{bmatrix}$$

The matrix B is diagonal, with elements $\tfrac{1}{2}(\mu^2, 1, \mu^2, 1, \mu^2, 1, \mu^2, -1, -1, -1)$. Unlike the Majorana case, we have here obtained a particular matrix representation for A^μ, B directly from the known form of the integral spin Lagrangians (4.1, 3). Any other representation is reached by a linear transformation on the Kemmer column symbol $\chi = S\chi'$. This replaces A^μ and B by $A^{\mu\prime} = \tilde{S}A^\mu S$, $B' = \tilde{S}BS$ thereby leaving the symmetry properties unaltered. The general results of Sec. II show directly that only *commutation* relations can occur. In particular, the singularity of A^0 only allows one to write the form (2.9b). For example, in spin 0, the last three components ϕ_i of χ do not enter in $A^0\chi$, being in fact determined from the constraint equations. The latter are defined in general to be those equations which are independent of time derivatives, and read in this case

$$\phi_i = \partial_i\phi - \partial H_I/\partial\phi_i \qquad (4.4)$$

Thus Eqs. (2.9b) represent the usual set of commutation relations between ϕ and $\phi_0 = \partial\mathcal{L}/\partial\dot{\phi}$. For spin 1, the four quantities ϕ_0, G_{ij} are missing from $A^0\chi$, the corresponding constraints being

$$\partial_i G_{0i} + \mu^2\phi_0 + \partial H_I/\partial\phi_0 = 0 \qquad (4.5a)$$

$$G_{ij} = \partial_i\phi_j - \partial_j\phi_i + \partial H_I/\partial G_{ij}. \qquad (4.5b)$$

If $\mu \neq 0$, Eqs. (4.5) may be solved for ϕ_0 and G_{ij}, again showing that Eqs. (2.9b) are the usual commutation relations between ϕ_i and G_{0i}. For the electromagnetic case, ϕ_0 no longer appears in Eq. (4.5a).[16] Instead, Eq. (4.5a) determines

[13] The theory (3.9) is invariant under chirality transformations, $\chi \to \gamma^5\chi$ just as is the normal massless Majorana case, however.

[14] The appearance of Bose commutation relations may also be understood in terms of derivations which assume invariance under TCP. In (3.9) the TCP operation reverses the sign of the free part of $\mathcal{L}$ (instead of leaving it invariant), and the connection is reversed.

[15] The other possible Hermitian form, $i[\phi, \partial_\mu\phi^\mu]$, is easily seen to lead either to no contribution to the dynamics when $\delta\phi_\mu$ commutes or to the inconsistency $\partial_\mu\phi = 0$ when $\delta\phi_\mu$ anticommutes.

[16] This is obvious in the first-order formulation of charged fields, whose $\mathcal{L}$, being linear in the derivatives, is therefore also linear in $(\partial_\mu - ieA_\mu)$. Hence $j^0 \equiv -\partial\mathcal{H}_I/\partial A_0$ is indeed independent of A_0. [Elimination of G_{ij} in j^0 by Eq. (4.5b) similarly cannot introduce any A_0 dependence since $\partial\mathcal{H}_I/\partial G_{ij}$ is independent of A_0.]

$\partial_i G_{0i}$ ($\equiv \nabla\cdot\mathbf{E}$) in terms of the other variables of the system. Thus, the longitudinal part of $\mathbf{E}$ is eliminated and the Bose quantization then follows in terms of the two independent transverse degrees of freedom of the photon.[17,18]

To summarize, the usual Bose quantization is valid for (massed or massless) neutral integral spin fields, while the nonoccurrence of symmetric A^μ forbids Fermi quantization.

V. CHARGED INTEGRAL SPIN

A charged field may be built up from two Hermitian fields by means of a 2×2 charge space. One simply defines $\chi \equiv (\chi_1, \chi_2)$, where $\chi_{1,2}$ are two independent Hermitian fields of the type considered in Sec. IV. Correspondingly, the dimensionality of A^μ, B is doubled in one of two possible ways. Thus, if a^μ, b are the 5×5 or 10×10 matrices of Sec. IV, then A^μ, B of the charged system are[19]

$$A^\mu = \begin{bmatrix} a^\mu & 0 \\ 0 & a^\mu \end{bmatrix}, \qquad B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \tag{5.1}$$

or

$$A^\mu = \begin{bmatrix} 0 & ia^\mu \\ -ia^\mu & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & ib \\ -ib & 0 \end{bmatrix}. \tag{5.2}$$

The significance of the two possibilities (5.1, 2) can easily be understood in terms of the usual charged fields $\phi \equiv (\chi_1 - i\chi_1)2^{-1/2}$ and ϕ^+. Thus, for spin zero, choice (5.1) gives (to within a divergence), for $\chi A^\mu\,\partial_\mu\chi - \partial_\mu\chi A^\mu\,\chi$, the form

$$\{\phi_\mu, \partial_\mu\phi^+\} + \{\phi_\mu^+, \partial_\mu\phi\} \tag{5.3a}$$

while choice (5.2) gives[20]

$$[\partial_\mu\phi^+, \phi_\mu] + [\phi_\mu^+, \partial_\mu\phi]. \tag{5.3b}$$

Any Hermitian form in terms of the ϕ fields may always be written as a linear combination of an anticommutator (5.3a) and a commutator (5.3b), and so the derivation of the connection for such a form is automatically covered in the cases (5.1) and (5.2).

The choice (5.1) leads to the correct Bose quantization only, since it preserves the antisymmetry of A^μ, which is all that was required in Sec. IV. The generator G_χ of the system with choice (5.1) is simply the sum $G_{\chi_1} + G_{\chi_2}$. It leads necessarily, by the techniques of the previous sections, to the standard Bose relation of each of the χ's with itself. No commutation relations between χ_1, and χ_2 can be deduced, however. If one wishes to interpret the system as a single charged field, then χ_1 and χ_2 are coupled through the electromagnetic interaction term. The current is, in fact, proportional to $\chi_2 a^\mu\chi_1 - \chi_1 a^\mu \chi_2$ by the usual gauge arguments. The choice $[\chi_1, \chi_2] = 0$ (which follows from the choice $]\delta\chi_1, \chi_2] = 0$) leads to the standard charged boson theory. The alternative, $\{\chi_1', \chi_2\} = 0$ makes the current vanish identically (since $\tilde{a}^\mu = -a^\mu$) and so it does not give rise to an electromagnetic interaction. While no inconsistency arises with this choice, the charge interpretation cannot be made; one has two electrically neutral fields, which may perhaps interact in other ways according to the structure of $\mathcal{H}_I$. (Conceivably, the choice $\{\chi_1', \chi_2\} = 0$ may be required for a particular $\mathcal{H}_I$ not to vanish.)[21] This would not represent a breakdown of the spin-statistics connection, since the two fields cannot then be combined into a single anticommuting one possessing a particle interpretation.

We now show that the extension to charge space according to Eq. (5.2), which can only lead to Fermi quantization since the product space A^μ is

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[17] For the first-order form of electrodynamics in the radiation gauge, see for example J. Schwinger, Phys. Rev. **115**, 721 (1959).

[18] The spin-2, zero-mass field may also be put into first-order form in terms of two transverse degrees of freedom [R. Arnowitt and S. Deser, Phys. Rev. **113**, 745 (1959)] so that here too Bose quantization may be deduced directly.

[19] That the extension to charge space can always be put into form (5.1) or (5.2) may be seen as follows. The independent 2×2 matrices are I, σ_i. The matrix I is symmetric and is the choice (5.1), while σ_2 is antisymmetric and yields (5.2). The other two matrices σ_1, σ_3 are symmetric; by a linear transformation in the 2-dimensional charge space, σ_1 can be reduced to σ_3 and hence does not yield an independent representation. Further, by means of a linear transformation in the original vector spaces of the Hermitian χ_1 and χ_2, one may show that the σ_3 representation is equivalent to that generated by I, that is, there exists a transformation T such that $\tilde{T}A^0T = -A^0$. (T simply interchanges canonical coordinates and momenta, e.g., sends $\phi\to\phi^0$, $\phi^0\to -\phi$ for the scalar field.) Hence in the product space

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{T} \end{bmatrix}(\sigma_3\otimes A^0)\begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{T} \end{bmatrix}\begin{bmatrix} A^0 & 0 \\ 0 & -A^0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} A^0 & 0 \\ 0 & A^0 \end{bmatrix} = I\otimes A^0$$

and σ_3 has been made equivalent to I. Finally, we note that unless the same charge matrix is used for both A^μ and B, the resulting charged fields, ϕ, ϕ^+, will not obey the appropriate field equation for the spin in question.

[20] Note the difference between the form (5.3b) for the charged field and that given in footnote 15 for a neutral field. Form (5.3b) clearly contributes only for anticommuting variations. The existence of the antisymmetric matrix B of (5.2) prevents the contradiction $\partial_\mu\phi = 0$ of the neutral case.

[21] A simple, if artificial, example is provided by $\mathcal{H}_I \sim \chi_2 s\chi_1 - \chi_1 s\chi_2$, $\tilde{s} = s$. It should be noted, however, that in general, one need not necessarily specify the relation between χ_1 and χ_2 if $\mathcal{H}_I$ does not require such a specification. For a discussion of this question, see for example G. Luders, Z. Naturforsch, **13a**, 254 (1958).

symmetric, is inconsistent. The generator here reads

$$G_\chi = \tfrac{1}{2}\int d^3r(\chi A^0\,\bar{\delta}\chi - \bar{\delta}\chi A^0\chi)$$
$$= \frac{i}{2}\int d^3r(\chi_1 a^0\,\bar{\delta}\chi_2 - \chi_2 a^0\,\bar{\delta}\chi_1 - \bar{\delta}\chi_1 a^0\chi_2 + \bar{\delta}\chi_2 a^0\chi_1). \qquad (5.4)$$

Since $\tilde{a}^0 = -a^0$, either $\{\bar{\delta}\chi_{1,2}, \chi_{2,1}\} = 0$ holds or else G_χ vanishes identically. Alternately, these anticommutation relations are necessary in order to obtain any field equations from $\mathfrak{L}$. Applying Eq. (2.3) with $\chi = \chi_1$ and $\bar{\delta}\chi_1 = 0$ one finds

$$\{(a^0\chi_1')_\alpha, (a^0\chi_1)_\beta\} = 0 \qquad (5.5)$$

and similarly for the independent variables $a^0\chi_2$ of the second field. This means, however, that the independent components of $\chi_{1,2}$, i.e., $(a^0\chi_{1,2})_\alpha$ square to zero, and being Hermitian, necessarily vanish. Thus, the usual Bose quantization of charged integral fields is alone permitted.

VI. CHARGED SPIN $\frac{1}{2}$

In this section, we apply our methods to the charged spin $\frac{1}{2}$ field and show that, just as in the massless neutral spin $\frac{1}{2}$ case, Bose quantization cannot be forbidden without further assumptions. In building up the charged field from two Majorana systems, there are the same two possibilities as in the integral spin case, namely, the usual direct product $I \otimes A^\mu$ (with mass term $mI \otimes B$) and the representation $\sigma_2 \otimes A^\mu$ (with mass term $m\sigma_2 \otimes B$). We begin with the case of nonvanishing mass, and establish first that A^μ must be symmetric and B antisymmetric, just as in the neutral case. This follows from the requirement that the equation for the charged field $\psi \equiv \chi_1 - i\chi_2$ have as its free particle term the Dirac form $(-i\gamma^\mu\,\partial_\mu + m)\psi$. Since, as can easily be checked, the $\chi_{1,2}$ have free particle parts $(A^\mu\,\partial_\mu + mB)\chi_{1,2}$ in either charge representation, the relation $\gamma^\mu = iB^{-1}A^\mu$ still holds. The symmetry properties $A^\mu = s^\mu$, $B = B^a$ (s^μ and B nonsingular) then follow from the Dirac algebra, as in the neutral case.

It is clear now that the choice $I \otimes s^\mu$ leads only to the usual correct Fermi quantization of the charged field (just as (5.1) did in the normal Bose case). The remaining possibility, $\sigma_2 \otimes s^\mu$, however, leads only to Bose statistics, since $\sigma_2 \otimes s^\mu$ is antisymmetric. In contrast to its analog (5.2), however, no algebraic inconsistencies arise from the "wrong" statistics but TCP is violated. This may seem surprising in view of the derivation in Sec. III for the massed Majorana case, where it was shown that the Dirac matrices γ^μ could not be built up from antisymmetric A^μ. That proof, however, depended on the fact that there are only sixteen 4×4 matrices available for a^μ, while the $\sigma_2 \otimes s^\mu$ space is now 8×8. The generator G_χ is here

$$G_\chi = i/2\int d^3r[\chi_1 s^0\,\bar{\delta}\chi_2 + \bar{\delta}\chi_2 s^0\chi_1 - \chi_2 s^0\,\bar{\delta}\chi_1 - \bar{\delta}\chi_1 s^0\chi_2] \qquad (6.1)$$

so that the symmetry of s^0 forces $[\bar{\delta}\chi_{1,2}, \chi_{1,2}] = 0$ to prevent the vanishing of G_χ and of the Lagrange equations. Applying Eq. (2.3) with $\chi = \chi_1$ and $\bar{\delta}\chi_1 = 0$ we find $[\chi_1', \chi_1] = 0$ and similarly for χ_2. Next, taking $\chi = \chi_1$, $\bar{\delta}\chi_2 = 0$ in Eq. (2.3), we find

$$\int d^3r'[\chi_1, \chi_2' s^0\,\bar{\delta}\chi_1'] = \tfrac{1}{2}\,\bar{\delta}\chi_1. \qquad (6.2)$$

In order to move $\bar{\delta}\chi_1$ to the same side in each term of the commutator in (6.2), we need the commutation relation of $\bar{\delta}\chi_1$ with χ_1 itself. The first possibility, $\{\bar{\delta}\chi_1, \chi_1\} = 0$, leads to an anticommutation relation between χ_1 and χ_2. However, at least the free part of the Hamiltonian,

$$H_0 = -i\int d^3r[\chi_1 s^i\,\partial_i\chi_2 - \chi_2 s^i\,\partial_i\chi_1] + im\int d^3r(\chi_1 b\chi_2 - \chi_2 b\chi_1), \qquad (6.3)$$

then vanishes (to within a c number), making the Heisenberg equations inconsistent with the Lagrange equations. The other possibility, $[\bar{\delta}\chi_1, \chi_1'] = 0$ leads to

$$[\chi_1, \chi_2'] = -i/2\;\delta^3(\mathbf{r} - \mathbf{r}') \qquad (6.4)$$

where we have chosen the representation $S^0 = iI$, since $\tilde{s}^0 = s^0$, $s^{0*} = -s^0$. In terms of the charged field, $\psi = \chi_1 - i\chi_2$, $\psi^+ = \chi_1 + i\chi_2$, we find

$$[\psi, \psi'] = 0 = [\psi^+, \psi^{+\prime}],$$
$$[\psi^+, \psi'] = \delta^3(\mathbf{r} - \mathbf{r}'). \qquad (6.5)$$

Here $\gamma^\mu = iB^{-1}s^\mu$ and $B^{-1} = \gamma^0$, which is a possible realization of the γ^μ as discussed in Sec. III. Further, the Heisenberg equations based on Eqs. (6.5) are consistent with the Lagrange equations.

The preceding discussion has thus led to Bose quantization of *this* representation of the charged spin $\frac{1}{2}$ field, free of any purely algebraic inconsistencies.[22] In the free case it is clearly sufficient

[22] It should be mentioned here that the Weyl 2-component neutrino theory may be quantized with Fermi or Bose statistics since it may be viewed as a special case of the Dirac neutrino obtained by projecting with $(1 + i\gamma^5)$. Alternately, one may see this from the fact that one may choose the Weyl Lagrangian in the form $[\psi^+, \sigma^\mu\partial_\mu\psi]$ or $i\{\psi^+, \sigma^\mu\partial_\mu\psi\}$. These two symmetrizations correspond to use of I or σ_2 in generating the charge space.

to demand positive energy, as Pauli[1] was led to do. More generally, however, this case may be eliminated in the interacting situation by invoking TCP invariance. For, it is easily seen that the free particle part of the Lagrangian is not invariant under TCP, since it is invariant under T and P but not under C. Also, as is well known, the current operator has a positive-definite charge density when Bose quantization is used, whereas it must change sign under TCP if TCP invariance is to hold.

Finally, we investigate the massless charged cases, which are obtained from the massless Majorana examples by use of I or σ_2. If A^μ is symmetric (so that an antisymmetric b must be used to form $\gamma^\mu = ib^{-1}A^\mu$) we have merely the $m=0$ limit of the massed cases treated above and the same conclusions apply. If, however, we consider A^μ antisymmetric [and hence b antisymmetric by Eq. (3.7)] we get the same sort of situation as in the non-TCP invariant neutral massless field, so that we are here building up the massless charged field from such massless neutral ones. Of the two possible charge representations, the σ_2 choice leads to a null theory: In this case (which is TCP invariant) the combined symmetry of $\sigma_2 \otimes \gamma^5\gamma^0\gamma^\mu$ implies Fermi quantization. We obtain then, the anticommutation relations $\{\psi'_\alpha, \psi^+_\beta\} = i/2\gamma^5_{\alpha\beta}\,\delta^3(\mathbf{r}-\mathbf{r}')$ for $\psi \equiv \chi_1 - i\chi_2$, whose spin trace implies that $\chi_1^2 + \chi_2^2 = 0$ and so that $\chi_1 = 0 = \chi_2$. On the other hand, the direct product $I \otimes \gamma^5\gamma^0\gamma^\mu$ clearly behaves just like the neutral case, namely we have $\mathcal{L} = \frac{1}{2}\{\psi^+, \gamma^5\gamma^0\gamma^\mu\partial\psi\}$, $[\psi', \psi^+] = -i/2\gamma^5\,\delta^3(\mathbf{r}-\mathbf{r}')$. Both TCP and positive-definiteness of the free Hamiltonian are violated, Bose quantization is not inconsistent, and so again the requirement of TCP invariance may be used to exclude this final case.

VII. COMMUTATION RELATIONS BETWEEN INDEPENDENT FIELDS

We discuss briefly the commutation relations among different Hermitian fields. For kinematically independent fields, i.e., systems which can be characterized by

$$\mathcal{L} = \tfrac{1}{2}\sum_i \{\chi_{(i)}A^\mu_{(i)}\,\partial_\mu\chi_{(i)} - \partial_\mu\chi_{(i)}A^\mu_{(i)}\chi_{(i)}\} - \mathcal{H},$$

the generator G_χ is also a sum of independent terms. Hence Eq. (2.5) yields no information about the relations between $\chi_{(i)}$ and $\chi_{(j)}$ $(i \neq j)$; that is, from

$$\int d^3r[\chi'_i, \chi_i A^0_i\,\bar{\delta}\chi_i] = \bar{\delta}_i\chi'_j = 0, \qquad (i \neq j) \qquad (7.1)$$

alone, one can obtain either $[\chi_i, \chi_j] = 0$ or $\{\chi_i, \chi'_j\} = 0$ depending on the choice of relations between $\bar{\delta}\chi_i$ and χ_j. The mechanism which imposes further restrictions in this formulation is the consistency requirement between G_χ and $\mathcal{H}$, i.e., between Heisenberg and Lagrange equations. For example, consider two uncoupled Bose hermitian fields χ_1, χ_2 so that $\mathcal{H} = \mathcal{H}_{01}(\chi_1) + \mathcal{H}_{02}(\chi_2) + \chi_2^3$. The Lagrange equations for χ_1 require no knowledge of the $\bar{\delta}\chi_1$, χ_2 relations. On the other hand, the Heisenberg equations, $\dot{\chi}_1 = i[H, \chi_1]$ involve $[\chi_1, \chi_2^3]$ so that χ_1 must commute with χ_2 for consistency. For general $\mathcal{H}$, the consistency requirement leads to commutation conditions identical to those previously given by Luders.[23]

For some interactions, the above consistency conditions do not restrict the relations between kinematically independent fields. However, making one or another choice (when either is allowed) can alter the physical interpretation of the theory. An example was given in the charged integral spin case, where it was seen that $\{\chi_1, \chi'_2\} = 0$ implied the vanishing of the current operator, and so that $\chi_{1,2}$ were two neutral Bose fields (possibly interacting), rather than the components of a single charged field $\phi = 2^{-1/2}(\chi_1 - i\chi_2)$. In this connection, one might note that Burgoyne's proof[3] led automatically to $[\phi', \phi^+] = 0$, $(\mathbf{r} \neq \mathbf{r}')$, whereas in our analysis this result holds only if the choice $[\chi_1, \chi'_2] = 0$ is taken. The difference lies in Burgoyne's interpretation of his assumption that all quantities either commute or anticommute: he treats the charged fields ϕ, ϕ^+ (rather than χ_1 and χ_2) as the entities which obey only one of the two possibilities (as is indeed the characteristic of the correct charged field). In our analysis, the relations between χ_1 and χ_2 may, *a priori*, differ from those of each field with itself, which makes the noncharged possibility also available.

VIII. CONCLUSIONS

In the present derivation of the spin-statistics connection, emphasis was put on the separation of the usual TCP invariance and vacuum state ($E \geq 0$) requirements from the conventional assumptions of local field theory. It was found that for all spin 0 and 1 fields and the neutral massed spin $\frac{1}{2}$ fields, the correct connection could be established purely from the algebraic form of the free particle part

[23] Luders[21] started from the requirement of local Heisenberg equations. We demand consistency between these and the (by assumption local) Lagrange equations, so that the two approaches are essentially the same. Within the framework of the axiomatic method, H. Araki [J. Math. Phys. **2**, 267 (1961)] has obtained the independent field relations.

of the Lagrangian, without recourse to the TCP or $E \geq 0$ requirements. Actually for these fields, it is the case that the *free parts* do satisfy TCP and $E \geq 0$. For the remaining spin $\frac{1}{2}$ cases, it was found that both connections occurred (as was the case in Pauli's original free field derivation). However, those cases for which the wrong statistics held were characterized by lack both of TCP invariance and of $E \geq 0$ in the free particle part of the Lagrangian. Hence these cases could be eliminated either by requiring TCP invariance or $E \geq 0$ *solely* for the free particle parts.

It is curious that the energy requirement is to be imposed on the free particle part of the energy rather than on the more physically meaningful total energy. This might appear more understandable if, as has been suggested, the sign of the total energy is always the sign of the kinetic energy. The significance of the alternative requirement, TCP invariance, seems somewhat more puzzling from the present approach. Invariance of a local field under TCP is a consequence of both proper Lorentz invariance and the *assumption* that the correct connection holds.[24] Consequently, *a priori* acceptance of TCP invariance is not so straightforward from the present point of view, and may be regarded as an empirical question. On the other hand, the fact that one need impose TCP only on the *free* particle part of the Lagrangian is reasonable. For, as we have seen, the latter requirement yields the correct connection which then implies TCP invariance for the total Lagrangian.[24]

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APPENDIX

A discussion of the postulates needed in the Schwinger action principle[4] is given here. We begin with some definitions and notations concerning unitary transformations on the basis vectors $|a'\rangle$ of a complete set of operators $\{A_i\}$ $(A_i \mid a'\rangle = a_i' \mid a'\rangle)$ in Hilbert space. Let an infinitesimal unitary transformation be $\mathfrak{U} \equiv 1 + iG$ where $G^+ = G$. If we denote the transformed ket by $\mid \bar{a}'\rangle \equiv \mathfrak{U}^{-1} \mid a'\rangle$, then the change in the ket due to the unitary transformation, $\delta \mid a'\rangle \equiv \mid \bar{a}'\rangle - \mid a'\rangle$, is given by $\delta \mid a'\rangle = -iG \mid a'\rangle$. The matrix elements of any operator B then change according to

$$\delta\langle a' |B| a''\rangle \equiv \langle \bar{a}' |B| \bar{a}''\rangle - \langle a' |B| a''\rangle.$$

If we define the operator $\delta_0 B$ by $\delta\langle a' \mid B \mid a''\rangle \equiv \langle a' \mid \delta_0 B \mid a''\rangle$, one has $\delta_0 B = -i[B, G]$. Thus the change of the matrix elements of an operator due to a change of basis can equivalently be represented by a change of the operators in the old basis.

If $\{A_i\}$ represents the complete set for the original basis $\mid a'\rangle$, then the complete set, $\{\bar{A}_i\}$ for the transformed basis $(\bar{A}_i \mid \bar{a}'\rangle = a_i' \mid \bar{a}'\rangle)$ is related to A by $\bar{A} = \mathfrak{U}^{-1} A \mathfrak{U} = A - \delta_0 A$. In accordance with the conventional physical interpretation of Hilbert space, one must associate, at any time t, a complete set of Hermitian operators $\{A_i(t)\}$ to a complete set of compatible observables. The simultaneous eigenkets of $A_i(t)$, i.e. $\mid a't\rangle$, form a basis which moves in time.[6] According to the probability interpretation, the bases at different times must be related by a unitary transformation. (The assumption of a positive-definite Hilbert-space metric is used here. This assumption is also used explicitly in some of the derivations given in text.) For the transformation representing an infinitesimal time translation (denoted by $G = G_t$) one has then that $\bar{A} = A(t + \delta t) = A(t) + \dot{A}\, \delta t$. Hence,

$$\dot{A}(t)\, \delta t = i[A(t), G_t(t)] \qquad \text{(A1a)}$$

since $\delta_0 A = -i[A, G_t]$ for this case. Equation (A1a) is the Heisenberg equation of motion. The corresponding basis vector equations of motion read: $\delta t(d \mid a't\rangle \mid dt) = -iG_t(t) \mid a't\rangle$.

The general variation of the transformation function to be considered here, $\delta\langle a_1 t_1 \mid a_2 t_2\rangle$, consists of changes of the bases due to their time motion plus variations due to changes of the complete set at a fixed time. Both variations are generated by unitary transformations so that

$$\delta\langle a_1 t_1 \mid a_2 t_2\rangle = i\langle a_1 t_1 \mid G_1(t_1) - G_2(t_2) \mid a_2 t_2\rangle. \qquad \text{(A2)}$$

Here $G(t)$ consists of two parts: one, the time translation generator G_t moving the system in time (keeping the same compatible set of measurables but at the displaced time), and a second part, G', generating the changes of bases possible at a fixed time (where the measurables are changed but the time is fixed). The change of the complete set generated by G' (i.e., $A \rightarrow \bar{A} = A - \delta_0 A$) is clearly given by

[24] See for example, G. Lüders, Ann. Phys. **2**, 1 (1957).

$$\delta_0 A = -i[A, G']. \qquad \text{(A1b)}$$

For a field system, any complete set, $\{A_i\}$, and hence the function G relating two complete sets, must depend only on the field variables χ_a. We now invoke the condition that we are dealing with a local field theory. This implies that complete sets $A(t)$, at time t, can be constructed from the field operators $\chi_a(t)$ at that t [i.e. $G'(t)$ depends only on $\chi_a(t)$]. Similarly, the future behavior of the kets and operators must be determined by the field variables at time t, i.e., $G_t(t)$ depends only on $\chi_a(t)$ (so that the dynamical laws be local in time). In general then, $G(t) = G[\chi(t)]$ is a local function, in time, of $\chi_a(t)$.

We now define the Hermitian operator δW_{12} by the equation

$$\delta\langle a_1t_1 \mid a_2t_2\rangle \equiv i\langle a_1t_1 \mid \delta W_{12} \mid a_2t_2\rangle \qquad \text{(A3)}$$

At this stage, δW_{12} depends only on variations at the end point times t_1 and t_2, according to Eq. (A2). However, we may divide the time interval (t_1, t_2) into many subintervals; the transformation function can then be represented by products of functions between the subintervals:

$$\langle a_1t_1 \mid a_2t_2\rangle = \sum \langle a_1t_1 \mid a_3t_3\rangle \times \langle a_3t_3 \mid a_4t_4\rangle \cdots \langle a_nt_n \mid a_2t_2\rangle. \qquad \text{(A4)}$$

In varying $\langle a_1t_1 \mid a_2t_2\rangle$ as expressed by the right-hand side of Eq. (A4), we can clearly make *arbitrary* unitary transformations on the bras and kets at the intermediate times, since such effects cancel out in the sum. In particular, one may consider interior variations in conflict with the actual time development. (For example, these intermediate variations may be generated by a G_t not proportional to the correct Hamiltonian.) On the other hand, the variation of Eq. (A4) leads to a sum of terms of the form

$$\sum_{a_m, a_{m+1}} \langle a_1t_1 \mid a_mt_m\rangle \times \delta(\langle a_mt_m \mid a_{m+1}t_{m+1}\rangle)\langle a_{m+1}t_{m+1} \mid a_2t_2\rangle = i\langle a_1t_1 \mid \delta W_{m,m+1} \mid a_2t_2\rangle \qquad \text{(A5)}$$

according to the definition (A3). This shows that δW_{12} may also be viewed as a sum of terms involving variations at the intermediate times, or, in the limit as the subintervals become infinitesimal in size, δW_{12} becomes a time integral between t_2 and t_1.

We now make the basic postulate that δW_{12} is the variation of a finite operator W_{12}, i.e., that the variations of the transformation function that we are considering are to be obtained by making appropriate variations[25] of W_{12}. (The nature of these variations will be found below). Since W_{12} is a time integral, we may write

$$W_{12} = \int_{t_2}^{t_1} d^4x\, \mathfrak{L}(x), \qquad \text{(A6)}$$

where $\mathfrak{L}$ must be a Hermitian operator. Comparing Eq. (A2) with Eq. (A3) gives

$$\delta \int_{t_2}^{t_1} d^4x\, \mathfrak{L}(x) = G_1(t_1) - G_2(t_2). \qquad \text{(A7)}$$

In Eq. (A7), variations of W_{12} can be made in the interior as well as at the end points, since arbitrary unitary transformations are allowed at interior time when varying the right-hand side of Eq. (A4) [as expressed in Eq. (A5)]. As $G(t)$ is a local function of $\chi_a(t)$ in time, Eq. (A7) represents a quantum Hamilton's principle for obtaining Lagrange equations of motion (this will be shown below).

We now make the further postulate that $\mathfrak{L}(x)$ has the form

$$\mathfrak{L}(x) = \tfrac{1}{2}(\chi A^\mu\, \partial_\mu\chi - \partial_\mu\chi A^\mu\chi) - \mathcal{H}(\chi) + \partial_\mu W^\mu(\chi), \qquad \text{(A8)}$$

where χ is a column symbol whose components, $\chi_a(x)$, are Hermitian field operators and A^μ are constant square matrices in the χ space. Hermiticity of $\mathfrak{L}$ is obtained by requiring that $A^{\mu+} = -A^\mu$, $\mathcal{H}^+ = \mathcal{H}$, $W^{\mu+} = W^\mu$. In order to have Lorentz invariance, we require that $\mathcal{H}$ be a scalar and W^μ a four-vector. Note that for invariance under the inhomogeneous Lorentz group to hold, neither $\mathcal{H}$ nor W^μ can depend explicitly on x^μ. The assumption of the form (A8) for $\mathfrak{L}(x)$ stems from the fact that the equations of motion for any Lorentz covariant field system involving fields of definite spin in local interaction may be obtained by varying a Lagrangian of the above "Kemmer-Dirac" type. The $\partial_\mu W^\mu$ term represents the usual freedom available of adding an arbitrary divergence to a Lagrangian without changing the equations of motion.

The action principle (A7) becomes well defined when the variations of W_{12} to be taken are specified. As discussed above, the variations of the transformation function $\langle a_1t_1 \mid a_2t_2\rangle$ under consideration involve changes in all the variables on which it

[25] We have restricted the analysis to variations corresponding to unitary transformations. The postulate can be shown to be also valid for certain other changes, such as source variations.[4]

depends, namely, the time[26] ($t \to t + \delta t(t)$) and the basis vectors (through the change in the complete set of operators). In general, irrespective of whether it is the variation of a finite operator W_{12}, the operator δW_{12} may be written as $\sum_m \delta W_{m,m+1}$ in the notation of Eq. (A5). Further, the infinitesimal operator $\delta W_{m,m+1}$ must have the form

$$\delta W_{m,m+1} = \int_{t_{m+1}}^{t_m} [Z_1(t;\chi(t))\,\delta t + Z_2(t;\chi(t))\,\Delta\chi]\,dt. \qquad \text{(A9a)}$$

The right-hand side expresses the most general form possible for the operator δW to yield the class of unitary transformations in the Hilbert space contained in the variation of the transformation function. Thus the term proportional to δt must be present, since changes $t \to t + \delta t$ are being considered, while the term proportional to the as yet undefined parameter $\Delta\chi$ has been included to account for the change of basis at fixed time (when δt is zero), i.e., for the change of the complete sets of operators. [In fact, even with a pure time translation, there is necessarily associated a change of the complete set $\{A(t)\} \to \{\bar{A} = A(t + \delta t)\}$, so that for this special case, we will see that $\Delta\chi$ is proportional to δt itself.] We now invoke the integrability postulate that there exists a $W_{12}(t;\chi)$, depending on the time and the field variables, so that its general variation consists in changing these arguments. Thus to have the form (A9a) arise from varying W_{12} we will assume that[27]

$$\delta W_{12} = W(t + \delta t, \chi + \delta'\chi) - W(t,\chi). \qquad \text{(A9b)}$$

The remaining problem, then, lies in finding the form of $\delta'\chi$ for a given variation of the transformation function $\langle a_1 t_1 \mid a_2 t_2\rangle$, (i.e., for a given infinitesimal unitary transformation).

The explicit variation of W_{12} reads

$$\delta W_{12} = \int d^4x[\{\delta\chi\cdot A^\mu\,\partial_\mu\chi - \partial_\mu\chi A^\mu\,\delta\chi - \delta\chi\,\partial\mathcal{H}/\partial\chi\} + \delta t(t)\,dT^{00}/dt] + \int d^3r[\tfrac{1}{2}(\chi A^0\,\delta\chi - \delta\chi A^0\chi) + \delta\chi(\partial W^0/\partial\chi) - \delta t(t)T^{00}]_{t_2}^{t_1} \qquad \text{(A9c)}$$

where $\delta\chi \equiv \delta'\chi + \dot{\chi}\,\delta t$ and

$$T^{00} = \tfrac{1}{2}(\partial_i\chi A^i\chi - \chi A^i\,\partial_i\chi) + \mathcal{H}.$$

The symbol $\delta\chi\,\partial\mathcal{H}/\partial\chi$ means $\mathcal{H}(\chi + \delta\chi) - \mathcal{H}(\chi)$. The condition that δW_{12} depend only upon end-point variations implies the vanishing of the first integral:

$$\int d^4x[\delta\chi A^\mu\,\partial_\mu\chi - \partial_\mu\chi A^\mu\,\delta\chi - \delta\chi\,\partial\mathcal{H}/\partial\chi] + \int dt\,\delta t\frac{d}{dt}H = 0, \qquad \text{(A10)}$$

where $H \equiv \int T^{00}\,d^3r$. The generator $G(t)$, which is obtained from the end-point terms of Eq. (A9c), according to (A7), is

$$G(t) = \int d^3r[\tfrac{1}{2}(\chi A^0\,\delta\chi - \delta\chi A^0\chi) + \delta\chi\,\partial w^0/\partial\chi] + \delta t H. \qquad \text{(A11)}$$

We consider first the case of no time motion, $\delta t = 0$, and obtain the generator

$$G'(t) = \int d^3r[\tfrac{1}{2}(\chi A^0\,\bar{\delta}\chi - \bar{\delta}\chi A^0\chi) + \bar{\delta}\chi\,\partial w^0/\partial\chi], \qquad \text{(A12)}$$

where $\bar{\delta}\chi$ denotes the value of $\delta'\chi$ for $\delta t = 0$. The generator G' must give rise to all possible fixed-time infinitesimal cannonical transformations. The form of G' clearly changes by changing W^0, so that W^0 must be regarded as an arbitrary function which generates the various possible bases. Further, the variation $\bar{\delta}\chi$ must necessarily be arbitrary at every space-time point.[28] This will allow one to have the freedom of generating different cannonical transformations in each of the independent mutually spacelike degrees of freedom of the field at time t. The Lagrange equations may now be obtained from Eq. (A10) by setting $\delta t = 0$. One has then that

$$(\bar{\delta}\chi A^\mu\,\partial_\mu\chi - \partial_\mu\chi A^\mu\,\bar{\delta}\chi) - \bar{\delta}\chi\,\partial\mathcal{H}/\partial\chi = 0. \qquad \text{(A13)}$$

In order to obtain explicit Lagrange equations of motion, some condition on the operator properties of $\bar{\delta}\chi$ is required since $\bar{\delta}\chi$ need not be a c number in a quantum theory. We postulate that $\bar{\delta}\chi_a$ either commutes or anticommutes with the field operators χ_b. With this assumption, one may move all the $\bar{\delta}\chi$ either to the left or to the right side and equate the coefficient of $\bar{\delta}\chi_a$ to zero. The condition of

[26] For simplicity, we are not varying the spatial coordinates x^i. Their variation would lead to the spatial translation operators (field momenta).

[27] Strictly speaking, the general form of δW is $W(t + \Delta t; \chi + \bar{\delta}\chi) - W(t;\chi)$ where Δt has a nonvanishing part even for $\delta t \to 0$, i.e., $\Delta t = \alpha(t)\delta t + \delta\beta$. The assumption made in (A 9b) is that $\delta\beta = 0$. Invariance under time translations implies that α is a constant, which may be set to unity by a choice of units.

[28] Unless $\bar{\delta}\chi$ is arbitrary in its time dependence [so that we may choose it proportional to $\delta(t)$], one would obtain from Eq. (A 10) a set of equations of motion nonlocal in time, instead of the local Lagrange equation (A 13). The postulate of time-locality of the dynamics forbids this. Lorentz-invariance then requires that $\bar{\delta}\chi$ also be arbitrary in its spatial dependence. Note also that $\delta'\chi$ is the $\Delta\chi$ of the general discussion of Eq. (8.9a) when Eq. (8.9b) has been postulated.

commutation or anticommutation on $\bar{\delta}\chi$ eventually leads to either commutation or anticommutation relations between the field operators themselves, and is an assumption conventionally made also in other derivations of the spin-statistics connection.[29]

Let us now ask for the part of G that generates pure time translations (with no change of the complete set at time t), i.e., G_t. This means that we must restrict our variations $\delta\chi$ to those appropriate for a time translation. The generator G_t gives rise to the Heisenberg equations of motion (A1a) which should give a well-defined statement of the future dynamical motion of the system. On the other hand, the arbitrary function W^0 enters in Eq. (A12) (while the dynamics described by the Lagrange equations (A13) is independent of W^μ). We conclude, therefore, that $\delta\chi$ must vanish for pure time motion, i.e.,

$$\delta'\chi = -\dot{\chi}\,\delta t = \delta_0\chi.$$

This leads to

$$G_t = -H\,\delta t \tag{A14}$$

and from (A1a) the usual Heisenberg equations of motion $\dot{A} = -i[A, H]$ where H is the conventional field Hamiltonian. Also, we note that imposing the the condition $\delta\chi = 0$ on Eq. (A10) gives rise to the consistent result[30] $dH/dt = 0$.

We turn now to the determination of the properties of G'. More precisely we will find the unitary transformations which G' generates and this information will yield the field commutation relations. In fact, a knowledge of the transformation generated by $G_\chi \equiv G'(W^0 = 0)$ is adequate to determine the properties of the general case. Thus, in the notation of Eq. (A1b), we write generally, $\delta_0\chi = \frac{1}{2}f\bar{\delta}\chi$ for the changes generated by G_χ, i.e.,

$$[\chi(\mathbf{r})G_\chi] = \tfrac{1}{2}if\,\bar{\delta}\chi(\mathbf{r}) \tag{A15a}$$

where f is at present unknown,[31] and may even be an operator function of χ. Thus G_χ generates the transformation

$$\chi \to \bar{\chi} = \chi - \tfrac{1}{2}f\,\delta\chi. \tag{A15b}$$

The commutation relations then follow from[32] Eq. (A15a), using the fact that $\bar{\delta}\chi(\mathbf{r})$ is an arbitrary function which either commutes or anticommutes with $\chi(\mathbf{r})$:

$$[\chi(\mathbf{r}), \chi(\mathbf{r}')A^0]_\mp = \tfrac{1}{2}if\,\delta^3(\mathbf{r}-\mathbf{r}') \tag{A16}$$

As discussed in Sec. II, the bracket in Eq. (A16) is a commutator (anticommutator) when A^0 is antisymmetric (symmetric). These complete commutation relations enable us to evaluate the commutator of χ with any function. In particular, one can find the analog of Eq. (A15b) for an arbitrary

$$G' = G_\chi + \int d^3r\,\bar{\delta}\chi(\partial W^0/\partial\chi)$$

where $\bar{\delta}\chi(\partial W^0/\partial\chi)$ is shorthand for $W^0(\chi+\bar{\delta}\chi) - W^0(\chi)$. In the discussion below, it will be convenient to consider W^0 as a function of $A^0\chi$ [i.e., $W^0 = W^0(A^0\chi)$] since these are the independent field variables (even when constraints exist). We restrict ourselves to W^0's which are even in the anticommuting field variables.[33] In this case, one finds by direct computation [using Eq. (A16)] that G' generates the change

$$\chi(\mathbf{r}) \to \bar{\chi}(\mathbf{r}) = \chi - \tfrac{1}{2}f\,\bar{\delta}\chi - \tfrac{1}{2}\,\partial^2W^0/\partial(A^0\chi)^2A^0f\,\bar{\delta}\chi, \tag{A17}$$

where $\partial^2W^0/\partial(A^0\chi)^2\frac{1}{2}iA^0f\bar{\delta}\chi$ stands for

$$\eta^{-1}[W^0(A^0\chi' + A^0\,\bar{\delta}\chi' + \tfrac{1}{2}iA^0\eta f) - W^0(A^0\chi' + A^0\,\bar{\delta}\chi') - W^0(A^0\chi' + \tfrac{1}{2}iA^0\eta f) + W^0(A^0\chi)] \tag{A18}$$

The η appearing in (A18) is a new infinitesimal which commutes or anticommutes with χ and $\bar{\delta}\chi$ [according to the sign in (A16)] and is to be moved to the left and canceled to obtain the explicit form. The operations in (A18) reduce to the usual definition of second derivative in the commuting situation, and provided the correct rule in the other case as well.

Information on the allowed type of f comes from the requirement that G' and G_t be mutually consistent, and that G_t be consistent with the time development as given by the Lagrange equations. To see this, we first consider $[H, G_\chi]$ which by (A15b) is

[29] Note that this restriction on $\bar{\delta}\chi$ means that in our discussion of G', there was no possibility of obtaining the infinity of different transformations by different choices of $\bar{\delta}\chi$ (due to its simple "c number" nature). The generality of different W^0's is therefore indeed necessary.

[30] With a more complete treatment involving space like surfaces and general coordinate variations (4) $\delta\chi^\mu$, this term gives rise to the local conservation laws $\partial_\nu T^{\mu\nu} = 0$.

[31] The operator f is necessarily coordinate independent: Translational invariance requires that $f(\mathbf{r}, \mathbf{r}') = f(\mathbf{r}-\mathbf{r}')$, while its $\delta^3(\mathbf{r}-\mathbf{r}')$ coefficient shows that only $f(0)$ enters. [See Eq. (A16) below.]

[32] We have assumed that the matrix A^0 is nonsingular (i.e., no constraints are present in the theory). When A^0 is singular, the discussion still follows in terms of the independent variables $A^0\chi$, using the result $[A^0\chi, A^0\chi']_\mp = (i/2)\,A^0f\delta^3(\mathbf{r}-\mathbf{r}')$.

[33] As noted above, the anticommuting case arises when A^0 is symmetric and so, as discussed in text, only for spin $\frac{1}{2}$. Lorentz invariance then requires that tensor quantities such as W^μ contain even powers of χ.

$$[H, G_\chi] = -i(H[\bar{\chi}] - H[\chi])$$
$$= -i(H[\chi - f/2\ \bar{\delta}\chi] - H[\chi])$$
$$= H\left[\chi + \frac{i}{2} f\ \bar{\delta}\chi\right] - H[\chi]. \quad \text{(A19)}$$

On the other hand, this evaluation must agree with the fact that H generates time translations. Thus

$$[H, G_\chi] = \int [H, \chi A^0]\ \bar{\delta}\chi\ d^3r + \int \chi A^0[H, \bar{\delta}\chi]\ d^3r. \quad \text{(A20)}$$

The commutator $[H, \bar{\delta}\chi]$ must vanish for all $\bar{\delta}\chi$. For, in the anticommuting case, one obtains from it a contribution due to parts of H which are odd in the anticommuting fields. However, such parts of H would also yield nonlocal contributions to the Heisenberg equations of motion (A19), and so contradict the local Lagrange equations. No such terms may appear in H, then.[33] The remaining term on the right in (A20) involves $\dot{\chi}A^0$ by Eq. (A1a), and so by the Lagrange equations (A13), we get

$$[H, G\chi] = H[\chi + \tfrac{1}{2}i\ \bar{\delta}\chi] - H[\chi]. \quad \text{(A21)}$$

Though the required consistency between Eqs. (A21) and (A19) strongly restricts the form of f, it does *not* permit one to conclude that $f = 1$ since H is not an arbitrary function of χ (and $\bar{\delta}\chi$ is not sufficiently arbitrary, since it only commutes or anticommutes with χ). Indeed, as has been noted by Wigner,[34] the consistency of the Lagrange and Heisenberg equations does not uniquely determine f for the simple case of the one-dimensional harmonic oscillator, but restricts it in the form $f = 1 + (2E_0 - 1) \exp [i\pi(H - E_0)]$ in units where $\hbar\omega = 1$. Here E_0 is the ground-state energy which is now arbitrary. In fact, in the one-dimensional case at least, such an indeterminacy exists in a large number of cases (for example, $V = \frac{1}{2}x^2 + \lambda x^3$). Further, $f = 1$ cannot in general be forced even when the same consistency requirement is imposed on the full generator G'. Thus, calculating $[H, G']$ as was done for $[H, G_\chi]$, using the results (A17, 18), one finds the analog of (A19) to be

$$[H, G'] = H[\chi + \tfrac{1}{2}if\ \bar{\delta}\chi + \tfrac{1}{2}i\ \partial^2 W^0/\partial(A^0\chi)^2 A^0 f\ \bar{\delta}\chi] - H[\chi] \quad \text{(A22)}$$

while (A21) is replaced by[35]

$$[H, G'] = \epsilon^{-1} \int d^3r[W(A^0\chi + A^0\ \delta\chi + \tfrac{1}{2}\epsilon\ \delta H/\bar{\delta}\chi) - W(A^0\chi + A^0\ \delta\chi) - W(A^0\chi + \tfrac{1}{2}\epsilon\ \delta H/\bar{\delta}\chi) + W(A^0\chi)]. \quad \text{(A23)}$$

where ϵ is a c number infinitesimal, and $\frac{1}{2}\delta H/\bar{\delta}\chi$ has been inserted for $A^0\dot{\chi}$ by the Lagrange equations. The symbol $\delta H/\bar{\delta}\chi$ is defined by

$$\delta H \equiv H(\chi + \bar{\delta}\chi) - H(\chi) = \int d^3r\ \bar{\delta}\chi(x)\ \delta H/\bar{\delta}\chi(x).$$

For all classical fields, the right sides of Eqs. (A22) and (A23) are trivially seen to be consistent only is the choice $f = 1$ is made. However, for quantum fields, it is not even obvious that these equations are consistent with $f = 1$. This is due to the fact that the order of operators in each equation is entirely different, so that the comparison can only be made after a large number of operator reorderings has been carried out [using the field commutation relations (A16)]. However, a somewhat tedious calculation (considering the general power series terms of H and W^0) establishes that, for $f = 1$, the results are indeed consistent.[36] Thus, the full generator G' is consistent with the possibility originally allowed by G_χ alone, that $f = 1$. The above results do not of course establish the necessity of $f = 1$. The Wigner example for f in the harmonic oscillator case turns to still yield consistent results between (A22) and (A23), although we have not investigated whether $f \neq 1$ is still possible for other potentials. Since it is impossible to deduce that f must be 1 for all systems,[9a] we add the consistent postulate that $f = 1$. Only in this way does the value of the fundamental commutator remain unchanged under change of basis. It is to be noted, however, that if one just assumes f to be a c number, the consistency between (A19) and (A21) is adequate to ensure that $f = 1$.

[34] E. P. Wigner, Phys. Rev. 77, 711 (1950).

[35] Equation (A23) is obtained by taking the time derivative of δW^0, and using the Lagrange equations to replace $A^0\dot{\chi}$ by $\delta H/\bar{\delta}\chi$.

[36] That is, all extra commutators arising in reordering one of the equations into the other's form cancel.